Advanced Nonlinear Studies **12** (2012), 767–782

The Grand Sobolev Homeomorphisms and Their Measurability Properties

Dedicated to Professor Antonio Ambrosetti

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Communicated by Andrea Malchiodi and Vittorio Coti Zelati

Abstract

We study the validity of the condition (N) of Lusin for homeomorphisms $f : \Omega \xrightarrow{\text{onto}} \Omega'$ under minimal assumptions on the integrability of Df. It turns out that the role of grand Sobolev spaces $W^{1,n)}$ and $W_b^{1,n)}$ is crucial. A discussion of bi-Sobolev maps in the plane and their connections with degenerate elliptic PDEs is provided.

2000 Mathematics Subject Classification. 46E35, 26B35.
Key words. Sobolev homeomorphism, Jacobian, Lusin condition

*This paper was performed while the second author was a Professor at the "Centro Interdisciplinare Linceo" of the Accademia Nazionale dei Lincei, Rome.

1 Introduction

Let f be a homeomorphism from a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) onto $\Omega' \subset \mathbb{R}^n$. We consider for f the following condition (N) of Lusin, $f \in (N)$

$$\text{if } E \subset \Omega \text{ with } |E| = 0, \text{ then } |f(E)| = 0 \tag{1.1}$$

where $|\cdot|$ denotes the Lebesgue measure. For a homeomorphism $f : \Omega \xrightarrow{\text{onto}} \Omega'$, condition (1.1) holds if and only if f maps measurable sets to measurable sets (see Section 2). Moreover, if f is differentiable at every point x of the Borel set $B \subset \Omega$ and $J_f(x)$ is the Jacobian determinant of f at x, then the *weak area formula* holds on B, that is

$$\int_B \eta(f(z))|J_f(z)|\mathrm{d}z \leq \int_{f(B)} \eta(w)\mathrm{d}w \tag{1.2}$$

for any nonnegative Borel-measurable function η on $\mathbb{R}^2$ (see Section 2). However, the (N) condition for such f and $B \subset \Omega$ is equivalent to the *area formula*

$$\int_B \eta(f(z))|J_f(z)|\mathrm{d}z = \int_{f(B)} \eta(w)\mathrm{d}w. \tag{1.3}$$

In this paper we address the following question.

What are the minimal integrability conditions on the partial derivatives of a Sobolev homeomorphism $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$ needed to guarantee that f satisfies (1.1)?

If the homeomorphism f satisfies the natural assumption $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$, then f verifies the (N)-condition (1.1). This is due to Reshetnjak [31] and it is a sharp result in the scale of $W^{1,p}(\Omega, \mathbb{R}^n)$-homeomorphisms thanks to an example of Ponomarev ([29], [30]) of a $W^{1,p}$-homeomorphisms $f : [0,1]^n \to [0,1]^n$, $p < n$ violating the (N)-condition. Another example has been constructed [22] of a homeomorphism $f \in \bigcap_{1 \leq p < n} W^{1,p}(\Omega, \mathbb{R}^n)$, satisfying the condition slightly below the natural one $|Df| \in L^n(\Omega)$,

$$\sup_{0<\varepsilon\leq n-1} \varepsilon \int_\Omega |Df|^{n-\varepsilon}\mathrm{d}x < \infty \tag{1.4}$$

and again f violates N-condition. When (1.4) occurs, we write $|Df| \in L^{n)}(\Omega)$, the grand Lebesgue space.

Condition (1.4) was introduced in [21] for mappings $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ (not necessarily homeomorphisms) in the study of the integrability of non negative Jacobians under minimal integrability assumptions for $|Df|$.

In [22] Kauhanen, Koskela and Malý proved that the Reshetnjak's sufficient condition $|Df| \in L^n(\Omega)$ for a homeomorphism $f : \Omega \xrightarrow{\text{onto}} \Omega'$ to satisfy condition (N) can be relaxed into the following one

$$\lim_{\varepsilon \to 0^+} \varepsilon \int_\Omega |Df|^{n-\varepsilon}\mathrm{d}x = 0 \tag{1.5}$$

(assuming $J_f(x) \geq 0$ a.e.), see also [23] and [9]. When (1.5) occurs, we write $|Df| \in L^{n)}_b(\Omega)$.

Actually in [22] the authors prove that, if the so called distributional Jacobian $\mathrm{Det}Df$ agrees with the pointwise Jacobian $J_f = \det Df$, then f verifies the (N) condition and then they use the fact, proved by L. Greco, that (1.5) is a sufficient condition under which $\operatorname{Det} Df$ agrees with J_f, if $J_f(x) \geq 0$ a.e. (see Section 2).

We simply mention here the fact that, for $n = 2$, the condition $Det\, Df = detDf$ is equivalent to the validity of integration by parts against the Jacobian J_f, that is, if $f = (u, v)$, then

$$\int_\Omega \varphi(u_x v_y - u_y v_x) = \int_\Omega u(\varphi_y v_x - v_y \varphi_x) = \int_\Omega v(\varphi_x u_y - \varphi_y u_x)$$

for all $\varphi \in C_0^1(\Omega)$, which is true for $f \in W_{loc}^{1,2}(\Omega, \mathbb{R}^2)$ but does not work when we only assume $f \in W_{loc}^{1,p}(\Omega, \mathbb{R}^2)$ for some $p < 2$, also in case of homeomorphisms.

Here for $n = 2$ we indicate another proof of Lemma 3.2 in [22] based on the following interesting approximation theorem (see [11]) which characterizes the closure of smooth homeomorphisms in the grand Sobolev space $W^{1,2)}$ (see Section 2). The question of diffeomorphic approximation of planar $W^{1,p}$- homeomorphisms, $p > 1$, has been recently settled by Iwaniec, Kovalev and Onninen (see [19]). Related approximation problems have also been treated by Daneri and Pratelli (see [6]).

Theorem 1.1 *Let Ω and Ω' be bounded domains of $\mathbb{R}^2$. If $f \in W^{1,1}(\Omega, \mathbb{R}^2) \cap \operatorname{Hom}(\Omega, \Omega')$ with $|Df| \in L_b^{2)}$ then, there exists a sequence f_j of diffeomorphisms such that*

$$f_j \to f \quad \text{in } C^0(\Omega; \Omega')$$

$$|Df_j| \to |Df| \quad \text{in } L^{2)}(\Omega)$$

$$\int_\Omega \varphi J_{f_j} \to \int_\Omega \varphi J_f \quad \forall \varphi \in C_0^1(\Omega).$$

In Section 3 we will consider Sobolev and Bounded Variation planar homemomorphisms and collect many interesting facts typical of dimension $n = 2$. In Section 4 we study planar bi-Sobolev maps, namely those homeomorphisms of the Sobolev class $W_{\mathrm{loc}}^{1,1}$ whose inverse is of the same Sobolev class, and their connections with homeomorphic solutions to degenerate Beltrami systems. Finally, in Section 5 we prove that a sufficient condition for a planar map to enjoy (N)-condition is that f belongs to the closure of diffeomorphisms in the grand Sobolev space $W^{1,2)}(\Omega, \mathbb{R}^2)$ (see Theorem 5.1).

2 Preliminaries

Let Ω and Ω' be bounded domains in $\mathbb{R}^n$ and let us denote by $Hom(\Omega, \Omega')$ the set of all homeomorphisms $f : \Omega \to \Omega' = f(\Omega)$. We say that the mapping $f : \Omega \to \mathbb{R}^n$ satisfies the *the Lusin condition* if

$$|E| = 0 \quad \Longrightarrow \quad |f(E)| = 0$$

for any measurable set $E \subset \Omega$. For the sake of completeness, let us give some auxiliary results together with their proofs.

Proposition 2.1 *If $f \in \operatorname{Hom}(\Omega, \Omega')$ then the Lusin condition holds for f if, and only if, f maps measurable sets to measurable sets.*

Proof. Let us assume that $f \in (N)$. If $A \subset \Omega$ is a measurable set, then there exists a Borel set B, $A \subset B \subset \Omega$ such that $|B \setminus A| = 0$. Then we have $|f(B) \setminus f(A)| = |f(B \setminus A)| = 0$, and hence $f(B) \setminus f(A)$ is measurable. Since $f(B)$ is a Borel set (because homeomorphisms map closed sets into closed sets), then $f(A)$ is also measurable. Conversely, suppose that $E \subset \Omega$ verifies $|E| = 0$ and $|f(E)| > 0$. Let $A' \subset f(E)$ be a non measurable set, then $f^{-1}(A') \subset E$ is a set of measure zero, hence $f^{-1}(A')$ is measurable and by assumption $A' = f(f^{-1}(A'))$ is measurable as well, which is a contradiction.

Proposition 2.2 *A homeomorphism $f : \Omega \to \Omega'$ satisfies the condition (N) iff $|f(E)| = 0$ whenever $E \subset\subset \Omega$ is a compact set with zero measure.*

Proof. If $E \subset \Omega$ satisfies $|E| = 0$, then there exists a Borel set $B \supset E$ such that $|B| = 0$. By contradiction if $|f(B)| > 0$, there exists a compact set $C' \subset f(B)$ such that $|C'| > 0$. On the other hand, since f is a homeomorphism, $f^{-1}(C')$ is compact and $|f^{-1}(C')| \leq |B| = 0$. This is not possible by assumption.

If $f \in \mathrm{Hom}(\Omega, \Omega')$ we decompose Ω as follows :

$$\Omega = \mathcal{R}_f \cup \mathcal{Z}_f \cup \mathcal{E}_f$$

where

$$\mathcal{R}_f = \{z \in \Omega : f \textit{ is differentiable at } z \textit{ and } J_f(z) \neq 0\} \tag{2.1}$$

$$\mathcal{Z}_f = \{z \in \Omega : f \textit{ is differentiable at } z \textit{ and } J_f(z) = 0\} \tag{2.2}$$

$$\mathcal{E}_f = \{z \in \Omega : f \textit{ is not differentiable at } z.\} \tag{2.3}$$

Differentiability is understood in the classical sense. Since f is continuous, these are Borel sets. Clearly we have

$$f(\mathcal{R}_f) = \mathcal{R}_{f^{-1}}. \tag{2.4}$$

Let us recall the *weak area formula* from Federer [[12] Theorem 3.1.8]. Let $B \subset \Omega$ be a Borel measurable set and assume that $f : \Omega \overset{\text{onto}}{\longrightarrow} \Omega'$ is a homeomorphism such that f is differentiable at every point of B, then for any $\eta : \mathbb{R}^n \to [0, +\infty[$ Borel measurable function we have

$$\int_B \eta(f(z))|J_f(z)|\mathrm{d}z \leq \int_{f(B)} \eta(w)\mathrm{d}w. \tag{2.5}$$

This follows from the area formula (1.3) which is valid for Lipschitz mappings and from the fact that the set of differentiability can be exhausted up to a set of zero measure by sets the restriction to which of f is Lipschitz [[12] Theorem 3.1.8]. Hence, for an a.e. differentiable homeomorphism on Ω we can decompose Ω into pairwise disjoint sets

$$\Omega = Z \cup \bigcup_{k=1}^{\infty} \Omega_k \tag{2.6}$$

such that $|Z| = 0$ and f_{Ω_k} is Lipschitz.

We note the following consequence of (2.5). If $B' \subset f(\Omega)$ is a Borel subset with $|B'| = 0$, then $J_f(x) = 0$ for a.e. $x \in f^{-1}(B')$. Indeed

$$\int_{f^{-1}(B')} |J_f(z)| \mathrm{d}z \leq \int_{B'} \mathrm{d}w = |B'| = 0.$$

For example, if f^{-1} is differentiable a.e. on $f(\Omega)$, then $J_f(x) = 0$ for a.e. $x \in f^{-1}(\mathcal{E}_{f^{-1}})$ where

$$\mathcal{E}_{f^{-1}} = \{z \in \Omega : f^{-1} \textit{ is not differentiable at } z\}.$$

We say that *the area formula* holds for f on B if (2.5) is valid as an equality, that is

$$\int_B \eta(f(z))|J_f(z)| \mathrm{d}z = \int_{f(B)} \eta(w) \mathrm{d}w \tag{2.7}$$

for all $\eta : \mathbb{R}^n \to [0, +\infty[$ Borel measurable function.

For a Sobolev homeomorphisms $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ (that is if the coordinate functions of f belong to the Sobolev space $W^{1,1}(\Omega)$ of L^1-functions $u : \Omega \to R$ whose gradient $|\nabla u|$ belongs to $L^1(\Omega)$) it is well known that there exists a set $\tilde{\Omega}$ of full measure such that the area formula holds for f on $\tilde{\Omega}$. Also, the area formula holds on each set on which the Lusin condition (N) is satisfied (this follows from the area formula for Lipschitz mappings, from the a.e. approximate differentiability of f [see [12] Theorem 3.1.4] and the already mentioned general property of a.e. differentiable functions [[12] Theorem 3.1.8] namely that Ω can be exhausted up to a set of measure zero by sets the restriction to which of f is Lipschitz continuous).

So, if we choose the Borel set $B = \mathcal{Z}_f$ as defined in (2.2) then by (1.3), we deduce

$$|f(\mathcal{Z}_f)| = 0$$

which is a weak version of the classical Sard lemma.

Let $f : \Omega \xrightarrow{\text{onto}} \Omega'$ be a homeomorphism. Then f maps every Borel set $B \subset \Omega$ onto a Borel set. Note that here we need to restrict ourselves to Borel sets B only since the homeomorphic image of a measurable set need not remain measurable.

In fact if f is a Cantor type homeomorphism $f : [0, 1] \to [0, 2]$ such that a zero set N_0, $|N_0| = 0$ is mapped to a positive set $P_0' = f(N_0)$, $|P_0'| > 0$ and E' is a non measurable set contained in P_0' (recall that every set of Lebesgue positive measure contains a non measurable subset) then $f^{-1}(E')$ is contained in the null set N_0 hence it is measurable.

The question of the differentiability in the classical sense of a homeomorphisms has a rather simple positive answer in the case $n = 2$ thanks to a classical Theorem of Gehring-Lehto (see [13], [24] and [26]).

Theorem 2.1 *Let Ω and Ω' be bounded domains in the plane and suppose that $f \in Hom(\Omega, \Omega')$ has finite partial derivatives a.e. in Ω, then f is differentiable a.e. in Ω.*

Remark 2.1 As a consequence, if $f = (u, v) : \Omega \subset \mathbb{R}^2 \xrightarrow{\text{onto}} \Omega' \subset \mathbb{R}^2$ is a Sobolev-homeomorphism, then f and f^{-1} are differentiable a.e. ([16]). The fairly well-known Theorem of Gehring-Lehto is one of the few facts from real analysis that carry geometric information up from the infinitesimal level. Its proof uses properties of the plane, in fact the Theorem at this stage of generality (f a BV-homeomorphism or f a $W^{1,1}$-homeomorphism) is false in higher dimension.

In the general case $n \geq 2$, the minimal integrability conditions on the partial derivatives of a Sobolev homeomorphism $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$ needed to guarantee a.e. differentiability have been found by J. Onninen [28], generalizing a classical result of Stein [33].

Namely, it turns out that if $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ and $|Df| \in L^{n-1,1}(\Omega)$ (where the Lorentz space $L^{p,1}(\Omega)$, $1 \leq p < \infty$ is defined as the class of all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^{p,1}(\Omega)} = p \int_0^\infty |\{z \in \Omega : |u(z)| > t\}|^{1/p} dt$$

is finite) then the homeomorphisms f and f^{-1} are differentiable a.e.

This is sharp after an example of a $W^{1,n-1}$-homeomorphism f ($n \geq 3$) which is bi-Sobolev (that is, $f^{-1} \in W^{1,1}$) and both f, f^{-1} are nowhere differentiable [5].

Let us prove the following useful result which generalizes Lemma 3.4 of [24].

Proposition 2.3 *If f is a Sobolev homeomorphism such that $J_f \geq 0$, then f^{-1} satisfies the condition (N), if and only if, $J_f(z) > 0$ for a.e. $z \in \Omega$.*

Proof. Suppose first that $f^{-1} \in (N)$ and denote by $\tilde{\Omega}$ a subset of Ω of full measure such that the area formula (2.7) with $B = \tilde{\Omega}$ holds true. Hence,

$$|f(\{z \in \tilde{\Omega} : J_f(z) = 0\})| = 0$$

and by (N) condition for f^{-1} we have

$$|\{z \in \Omega : J_f(z) = 0\}| = |\{z \in \tilde{\Omega} : J_f(z) = 0\} \cup (\Omega \setminus \tilde{\Omega})| = 0.$$

Conversely, suppose $J_f(z) > 0$ a.e. and let us prove that $f^{-1} \in (N)$. Assuming by contradiction that there exists $|N_0'| = 0$, $N_0' \subset \Omega'$ with $|f^{-1}(N_0')| > 0$, then we have

$$\int_{f^{-1}(N_0')} J_f \leq |f(f^{-1}(N_0'))| = |N_0'| = 0.$$

Hence $J_f = 0$ on the positive set $f^{-1}(N_0') \subset \Omega$ and this is a contradiction.

Let us prove the following simple characterization of the (N) condition for f:

Proposition 2.4 *If $f : \Omega \xrightarrow{onto} \Omega'$ is a Sobolev homemorphism, $J_f \geq 0$ and*

$$\int_B J_f = |f(B)|$$

for any Borel set $B \subset \Omega$ then, $f \in (N)$ on every Borel set $B \subset \Omega$.

Proof. By contradiction, there exists a subset $E \subset B : |E| = 0$ and $|f(E)| > 0$ then

$$\int_B J_f = \int_{B \setminus E} J_f \leq |f(B \setminus E)| = |f(B)| - |f(E)| < |f(B)|$$

and this is a contradiction.

An interesting application of condition (N) is the following result on the inverse of an a.e. differentiable homeomorphism which in the plane has an interesting counterpart (see Remark 2.1).

Proposition 2.5 *Let $f \in Hom(\Omega, \Omega')$ be differentiable a.e.. If f verifies the condition (N), then the inverse f^{-1} is differentiable a.e..*

Proof. We notice that the area formula (1.3) holds on each set on which f satisfies the (N)-condition; in particular it holds on $\mathcal{R}_f \cup \mathcal{Z}_f$ that is the set where f is differentiable (see Proposition **??**) :

$$\int_{\mathcal{R}_f \cup \mathcal{Z}_f} \eta(f(z))|J_f(z)| \mathrm{d}z = \int_{f(\mathcal{R}_f \cup \mathcal{Z}_f)} \eta(w) \mathrm{d}w. \tag{2.8}$$

In particular, we have the following version of Sard Lemma

$$|f(\mathcal{Z}_f)| = 0. \tag{2.9}$$

Since f is differentiable a.e., $\mathcal{E}_f$ has measure zero and by condition (N) $f(\mathcal{E}_f)$ has measure zero. We note that f^{-1} is differentiable in $f(\mathcal{R}_f)$ which is a subset of full measure of $f(\Omega)$; indeed,

$$f(\Omega) \setminus f(\mathcal{R}_f) = f(\mathcal{Z}_f) \cup f(\mathcal{E}_f)$$

has measure zero by (2.9) and condition (N).

By $A \triangle B$, we denote the set $(A \cup B) \setminus (A \cap B)$ and by $A = B$ a.e. we mean $|A \triangle B| = 0$.

Proposition 2.6 *Let $f \in \mathrm{Hom}(\Omega, \Omega')$ and assume that f and f^{-1} are differentiable a.e. and both verify condition (N) then, f essentially maps $\mathcal{E}_f$ to $\mathcal{Z}_{f^{-1}}$ and f^{-1} maps $\mathcal{E}_{f^{-1}}$ to $\mathcal{Z}_f$ in the sense that:*

$$|f(\mathcal{E}_f) \triangle \mathcal{Z}_{f^{-1}}| = |f(\mathcal{E}_{f^{-1}}) \triangle \mathcal{Z}_f| = 0.$$

Proof. Following the same arguments of Proposition 2.5,

$$|\mathcal{E}_f| = |\mathcal{E}_{f^{-1}}| = 0$$

and by Sard Lemma

$$|f(\mathcal{Z}_f)| = |f^{-1}(\mathcal{Z}_{f^{-1}})| = 0.$$

Then using the relation

$$f(\mathcal{Z}_f) \cup f(\mathcal{E}_f) = f(\mathcal{Z}_f \cup \mathcal{E}_f) = \mathcal{Z}_{f^{-1}} \cup \mathcal{E}_{f^{-1}}$$

yields

$$f(\mathcal{E}_f) \triangle \mathcal{Z}_{f^{-1}} = \Big(f(\mathcal{E}_f) \setminus \mathcal{Z}_{f^{-1}}\Big) \cup \Big(\mathcal{Z}_{f^{-1}} \setminus f(\mathcal{E}_f)\Big) \subset \mathcal{E}_{f^{-1}} \cup f(\mathcal{Z}_f)$$

we have

$$|f(\mathcal{E}_f) \triangle \mathcal{Z}_{f^{-1}}| = 0$$

and similarly

$$|f^{-1}(\mathcal{E}_{f^{-1}}) \triangle \mathcal{Z}_f| = 0.$$

3 Sobolev and BV-homeomorphisms in the plane

In this section we discuss some results about the regularity of the inverse of Sobolev or BV-homeomorphism in the plane. These results are of particular importance as Sobolev and BV spaces are commonly used as initial spaces for existence problems in Calculus of Variations.

Let Ω and Ω' be bounded open domains in the plane. We say that $f \in Hom(\Omega, \Omega')$ is a *Sobolev homeomorphism* if $f \in W^{1,1}_{loc}(\Omega; \mathbb{R}^2)$. Here, for $p \geq 1$, $W^{1,p}_{loc}$ consists of all locally p-integrable mappings of Ω into $\mathbb{R}^2$ whose coordinate functions have p-integrable distributional derivatives.

Similarly, we say that $f \in Hom(\Omega, \Omega')$ is a *BV (or bounded variation) homeomorphism* if $f \in BV(\Omega, \mathbb{R}^2)$ i.e. $f = (u, v)$ is a locally integrable map of Ω into $\mathbb{R}^2$, whose coordinate functions u, v have first order distributional derivatives which can be identified with measures with finite total variation in Ω. This means that there are Radon (signed) measures μ_1, μ_2, ν_1, ν_2 defined on Ω, so that $|\mu_i|(\Omega) < \infty$ and $|\nu_i|(\Omega) < \infty$ for $i = 1, 2$ and

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = - \int_\Omega \varphi d\mu_i$$

$$\int_\Omega v \frac{\partial \varphi}{\partial x_i} dx = - \int_\Omega \varphi d\nu_i$$

for all $\varphi \in C^1_0(\Omega)$. Further, $f \in BV_{loc}(\Omega, \mathbb{R}^2)$ requires $f \in BV(\tilde{\Omega}, \mathbb{R}^2)$, for each subdomain $\tilde{\Omega} \subset\subset \Omega$.

In [16], the authors proved that if $f : \Omega \overset{\text{onto}}{\longrightarrow} \Omega'$ is a BV_{loc} homeomorphism then so does its inverse $f^{-1} : \Omega' \overset{\text{onto}}{\longrightarrow} \Omega$. This result fails in the category of Sobolev homeomorphisms: if $f \in W^{1,1}$ it is not automatic that $f^{-1} \in W^{1,1}$. Moreover the result fails for $n \geq 3$, and in [16] an example is produced of homeomorphic $f \in W^{1,n-1-\varepsilon}(] - 1, 1[^n, \mathbb{R}^n)$ for which $f^{-1} BV_{loc}(f(\Omega), \mathbb{R}^n)$.

In [10] the following identities for total variations have been recently proved for the planar BV-homeomorphism $f = (u, v)$, whose inverse is $f^{-1} = (x, y)$:

$$\begin{aligned} |\nabla x|(\Omega') &= \left| \frac{\partial f}{\partial y} \right| (\Omega) \\ |\nabla y|(\Omega') &= \left| \frac{\partial f}{\partial x} \right| (\Omega). \end{aligned} \tag{3.1}$$

Let us state some important facts for BV or Sobolev homeomorphisms in the plane:

- If f is a BV_{loc}-homeomorphism, then f^{-1} is BV_{loc}. Moreover f and f^{-1} are differentiable a.e. in the classical sense ([13], [26]). In particular, the inverse of a $W^{1,1}_{loc}$-homeomorphism $f : \Omega \overset{\text{onto}}{\longrightarrow} \Omega'$ is in BV_{loc}. A sufficient condition in order that f^{-1} belongs actually to $W^{1,1}_{loc}$ is that $f^{-1} \in (N)$ according to the following:

Theorem 3.1 *Let $f : \Omega \overset{onto}{\longrightarrow} \Omega'$ be a planar $W^{1,1}$-homeomorphism and assume that f^{-1} satisfies the (N)-condition. Then $f^{-1} \in W^{1,1}_{loc}(\Omega', \mathbb{R}^2)$.*

Proof. Indicating $f = (u, v)$ and $f^{-1} = (x, y)$, it is sufficient to prove that for any compact set $E' \subset \Omega'$ such that $|E'| = 0$ we have

$$|\nabla x|(E') = 0 \qquad |\nabla y|(E') = 0$$

where $|\nabla x|(E')$ and $|\nabla y|(E')$ denote the total variations of the coordinate functions x and y of f^{-1} evaluated at E'. In fact if $F' \subset \Omega'$ is an arbitrary measurable null set $|F'| = 0$, then there exists a Borel set $B' \supset F'$ such that $|B'| = 0$. We claim that

$$|\nabla x|(F') \leq |\nabla x|(B') = 0.$$

Otherwise, if $|\nabla x|(B') > 0$ there would be a compact null set $E' \subset B'$ such that $|\nabla x|(E') > 0$ by the regularity of Borel measure and this is a contradiction.

Let us use the following equalities from [7]

$$\begin{aligned} |\nabla x|(A') &= \int_{f^{-1}(A')} \left|\frac{\partial f}{\partial y}\right| \\ |\nabla y|(A') &= \int_{f^{-1}(A')} \left|\frac{\partial f}{\partial x}\right| \end{aligned} \tag{3.2}$$

Let us fix the compact null subset $E' \subset \Omega'$ and define $A' = \Omega' \setminus E'$. By (N)-property for f^{-1} we have $|f^{-1}(E')| = 0$, hence

$$|f^{-1}(A')| = |f^{-1}(\Omega') \setminus f^{-1}(E')| = |f^{-1}(\Omega')|$$

Using twice (3.2) we obtain

$$|\nabla x|(A') = \int_{f^{-1}(A')} \left|\frac{\partial f}{\partial y}\right| = \int_{f^{-1}(\Omega')} \left|\frac{\partial f}{\partial y}\right| = |\nabla x|(\Omega')$$

and similarly

$$|\nabla y|(A') = |\nabla y|(\Omega')$$

by additivity properties of measures we arrive at

$$|\nabla x|(E') = |\nabla x|(\Omega' \setminus A') = 0$$

$$|\nabla y|(E') = |\nabla y|(\Omega' \setminus A') = 0$$

hence $|\nabla x|, |\nabla y| \in L^1(\Omega)$ that completes the proof.

However, condition (N) for f^{-1} is not necessary to have $f^{-1} \in W^{1,1}$. See section 4 where examples of bi-Sobolev maps without (N)-condition are shown. Further interesting facts are the following:

- If f is a BV-homeomorphism and $\tilde{J}_f$ denotes the Jacobian determinant of the absolutely continous part $\tilde{Df}$ of the differential Df, then $\tilde{J}_f \geq 0$ a.e. or $\tilde{J}_f \leq 0$ a.e. (see [17]).
- If f is Sobolev homeomorphism, then the integrability of the Jacobian is automatic and

  $$\int_\Omega |J_f| \leq |f(\Omega)|. \tag{3.3}$$

  Moreover, equality occurs in (3.3) if $f \in (N)$ (see Proposition 2.4 and [24]).
- If f is a Sobolev homeomorphism then $J_f \geq 0$ or $J_f \leq 0$ a.e. (the present result is true also for $n = 3$ but fails for $W^{1,1}$-homeomorphism for $n \geq 4$).

Proposition 2.3 has a counterpart valid for planar BV-homeomorphisms

Proposition 3.1 *Let* $f \in \mathrm{Hom}(\Omega, \Omega') \cap BV(\Omega, \mathbb{R}^2)$ *then the weak area formula*

$$\int_B \eta(f(z))\tilde{J}_f(z))\mathrm{d}z \leq \int_{f(B)} \eta(w)\mathrm{d}w \tag{3.4}$$

holds for $B \subset \Omega$ *Borel set and* $\eta : \mathbb{R}^2 \to [0, +\infty)$ *Borel measurable. Moreover*

$$\tilde{J}_f(z) > 0 \ a.e. \qquad \Longleftrightarrow \qquad f^{-1} \in (N)$$

Proof. Formula (3.4) is due to [4]. Assume $\tilde{J}_f > 0$ and by contradiction that there exists $N_0' \subset \Omega'$ such that $|N_0'| = 0$ and $|f^{-1}(N_0')| > 0$. Applying (3.4) with $\eta = 1$ on the Borel set N_0' and $\eta = 0$ on $\mathbb{R}^2 \setminus N_0'$ we obtain:

$$0 < \int_{f^{-1}(N_0')} \tilde{J}_f(x)\mathrm{d}x \leq |N_0'| = 0$$

which is impossible.

Conversely, if $f^{-1} \in (N)$ then (3.4) holds as an equality. Let

$$\tilde{\mathcal{Z}}_f = \{z \in \Omega : f \text{ is differentiable at } z \text{ and } \tilde{J}_f(z) = 0\}$$

and

$$0 = \int_{\tilde{\mathcal{Z}}_f} \tilde{J}_f(z)\,\mathrm{d}z \ = \ |f(\tilde{\mathcal{Z}}_f)|$$

by (N) condition on f^{-1} we deduce that $|\tilde{\mathcal{Z}}_f| = |f^{-1}(f(\tilde{\mathcal{Z}}_f))| = 0$, hence $\tilde{J}_f(z) > 0$ a.e..

4 Bi-Sobolev Mappings

In [18] a particularly useful class of homeomorphisms which lie between BV homeomorphisms and bi-Lipschitz mappings was introduced, namely the *bi-Sobolev mappings*

Definition 4.1 *The homeomorphism* $f : \Omega \subset \mathbb{R}^n \xrightarrow{onto} \Omega' \subset \mathbb{R}^n$ *is a bi-Sobolev map if* f *and* f^{-1} *are Sobolev homeomorphisms.*

The case $n = 2$ is quite special, in fact the following theorem holds (see [18])

Theorem 4.1 *To each bi-Sobolev mapping* $f : \Omega \xrightarrow{onto} \Omega'$ $f = (u, v)$ *there corresponds a measurable function* $A = A(z)$ *valued symmetric matrices with* $\det A(z) = 1$ *a.e. such that for a.e.* $z \in \Omega$ *and* $\forall \xi \in \mathbb{R}^2$

$$K(z)^{-1}|\xi|^2 \leq \langle A(z)\xi, \xi\rangle \leq K(z)|\xi|^2$$

where

$$K(z) = \begin{cases} \dfrac{|Df(z)|^2}{J_f(x)} & \text{if} \quad J_f(z) \neq 0 \\ 1 & \text{otherwise}. \end{cases}$$

The coordinate functions u, v of f are very weak solutions to the equation

$$\operatorname{div} A(z)\nabla u = 0$$

$$\operatorname{div} A(z)\nabla v = 0$$

with finite energy, i.e.

$$\int_\Omega \langle A(z)\nabla u, \nabla u\rangle = \int_\Omega \langle A(z)\nabla v, \nabla v\rangle \leq |f(\Omega)|$$

Moreover u and v have the same critical sets:

$$\{z \in \Omega : \nabla u(z) = 0\} = \{z \in \Omega : \nabla v(z) = 0\} \ \textit{a.e.}$$

A similar result is obtained in the setting of bi–ACL homeomorphisms (see [27]).

A sufficient condition that a Sobolev homeomorphism is a bi-Sobolev map is contained in the following (see [15]).

Theorem 4.2 *Let $f : \Omega \subset \mathbb{R}^2 \xrightarrow{onto} \Omega' \subset \mathbb{R}^2$ a Sobolev homeomorphism satisfying the condition*

$$|\mathcal{Z}_f| = 0 \tag{4.1}$$

on the critical set (2.2)*, then f is a bi-Sobolev map and*

$$\int_\Omega |Df| \mathrm{d}z = \int_{\Omega'} |Df^{-1}| \mathrm{d}w \tag{4.2}$$

We emphasize that condition (4.1) is not necessary for f to be a bi-Sobolev map. It can happen that bi-Sobolev maps have positive sets of critical points (see example of [30]).

Notice also that bi-Sobolev maps escape the pathological equality

$$|\mathcal{Z}_f| = |\Omega| \tag{4.3}$$

(see [18] and [8]).

Let us preliminarly prove the following

Lemma 4.1 *Let $f : \Omega \subset \mathbb{R}^2 \xrightarrow{onto} \Omega' \subset \mathbb{R}^2$ be a bi- Sobolev map with $J_f \geq 0$ a.e. in Ω. Then, the following conditions are equivalent each other*

$$J_f = 0 \ \textit{a.e. in}\ \Omega \tag{4.4}$$

$$\exists\, N_0 \subset \Omega, |N_0| = 0 \ \textit{such that}\ |f(N_0)| = |\Omega'| \tag{4.5}$$

$$\exists\, N'_0 \subset \Omega', |N'_0| = 0 \ \textit{such that}\ |f^{-1}(N'_0)| = |\Omega| \tag{4.6}$$

$$J_{f^{-1}} = 0 \ \textit{a.e. in}\ \Omega'. \tag{4.7}$$

Proof. (4.4) $\Longrightarrow$ (4.5). By area formula for Sobolev homeomorphisms, there exists $N_0 \subset \Omega$ with zero measure such that

$$\int_{\Omega \setminus N_0} J_f(z)dz = |f(\Omega \setminus N_0)|.$$

Then, by (4.4)

$$0 = |f(\Omega \setminus N_0)| = |\Omega' \setminus f(N_0)|$$

and hence (4.5) holds true.

(4.5) $\Longrightarrow$ (4.6). Define $N_0' = \Omega \setminus f(N_0)$ then obviously $|N_0'| = 0$. Moreover,

$$|f^{-1}(N_0')| = |f^{-1}(f(\Omega \setminus N_0))| = |\Omega \setminus N_0| = |\Omega|,$$

i.e. (4.6) holds true.

(4.6) $\Longrightarrow$ (4.7). By area formula

$$\int_{\Omega' \setminus N_0'} J_{f^{-1}}(w)dw \leq |f^{-1}(\Omega' \setminus N_0')| = |\Omega \setminus f^{-1}(N_0')| = 0$$

hence $\Omega' \setminus N_0' \subset \mathcal{Z}_{f^{-1}}$ and $|\Omega' \setminus \mathcal{Z}_{f^{-1}}| \leq |N_0'| = 0$ and (4.7) follows.

At this point we have proved also that (4.4) $\Longrightarrow$ (4.7), hence the implication (4.7) $\Longrightarrow$ (4.4) follows by symmetry and the Lemma is proved.

Theorem 4.3 *If* $f = (u, v) : \Omega \overset{onto}{\longrightarrow} \Omega'$ *is a bi-Sobolev map then*

$$|C_u| = |\{z \in \Omega : \ f \text{ is differentiable at } z \text{ and } |\nabla u(z)| = 0\}| =$$

$$|C_v| = |\{z \in \Omega : f \text{ is differentiable at } z \text{ and } |\nabla v(z)| = 0\}| =$$

$$= |\mathcal{Z}_f| < |\Omega|.$$

Proof. It is enough to show that $|\mathcal{Z}_f| < |\Omega|$ because everything else is proved in ([18], Theorem 2).

Suppose by contradiction that the pathological equality $|\mathcal{Z}_f| = |\Omega|$ holds true. Then we will deduce that the gradient of the inverse $f^{-1} = (x, y)$ is not absolutely continuous with respect to Lebesgue measure, that is $f^{-1} \notin W^{1,1}_{loc}$, which gives us a contradiction.

If $|\mathcal{Z}_f| = |\Omega|$, Lemma 4.1 implies that there exists $N_0' \subset \Omega'$ such that $|N_0'| = 0$ and $|f^{-1}(N_0')| = |\Omega|$. By regularity properties of Lebesgue measure there exists a closed set $C \subset f^{-1}(N_0')$ such that $|C| > 0$. Define $C' = f(C)$ and notice that it is a closed subset of N_0' with zero measure: $|C'| = 0$. We can apply to the open set $A' = \Omega' \setminus C'$ formula (3.2) and get

$$|\nabla x|(C') = |\nabla x|(\Omega') - |\nabla x|(A') = \int_{f^{-1}(\Omega')} \left|\frac{\partial f}{\partial y}\right| - \int_{f^{-1}(A')} \left|\frac{\partial f}{\partial y}\right| = \int_{f^{-1}(C')} \left|\frac{\partial f}{\partial y}\right| > 0$$

because $|C'| = 0$ and $|f^{-1}(C')| > 0$. This means that ∇x is not absolutely continuous with respect to Lebesgue measure and $f^{-1} \notin W^{1,1}_{loc}$ which gives the contradiction.

Remark 4.1 We observe that for a bi-Sobolev map $f : \Omega \xrightarrow{\text{onto}} \Omega'$ we have $|\mathcal{R}_f| > 0$ and $|\mathcal{R}_{f^{-1}}| > 0$. This follows by Gehring– Lehto Theorem which implies:

$$|\mathcal{Z}_f \cup \mathcal{R}_f| = |\Omega|$$

with

$$|\mathcal{Z}_{f^{-1}} \cup \mathcal{R}_{f^{-1}}| = |\Omega'|$$

and by the two inequalities:

$$|\mathcal{Z}_f| < |\Omega|$$

$$|\mathcal{Z}_{f^{-1}}| < |\Omega'|$$

deduced by Theorem 4.3.

5 Grand Sobolev Spaces and the condition (N)

Let us start with the so-called grand Lebesgue space $L^{q)}(\Omega)$ $q > 1$, $\Omega \subset \mathbb{R}^n$ a bounded domain (see [20], [21]). By definition it consists of measurable functions

$$u \in \bigcap_{1 \le s < q} L^s(\Omega)$$

such that

$$\|u\|_{L^{q)}} = \sup_{0<\varepsilon\le q-1} \left[\varepsilon \int_\Omega |u|^{q-\varepsilon} \mathrm{d}x\right]^{\frac{1}{q-\varepsilon}} < \infty.$$

Then $L^{q)}(\Omega)$ is a Banach space for $q > 1$, satisfying

$$L^{q,\infty}(\Omega) \subset L^{q)}(\Omega) \subset \bigcap_{1 \le s < q} L^s(\Omega)$$

where $L^{q,\infty}(\Omega)$ is the Marcinkiewicz weak L^q-space. We denote by $L^{q)}_b(\Omega)$ the closure of $L^\infty(\Omega)$ in $L^{q)}(\Omega)$. It is well known that $u \in L^q_b(\Omega)$ if and only if

$$\lim_{\varepsilon\to 0} \varepsilon \int_\Omega |u|^{q-\varepsilon} \mathrm{d}x = 0$$

(see [14]). The grand Sobolev space $1 < q < \infty$ $W^{1,q)}(\Omega)$ is the set of measurable functions $u \in L^{q)}(\Omega)$ such that $\||\nabla u|\|_{L^{q)}}(\Omega) < \infty$, equipped with the norm

$$\|u\|_{W^{1,q)}(\Omega)} = \|u\|_{L^{q)}\Omega)} + \||\nabla u|\|_{L^{q)}(\Omega)}.$$

The space $W^{1,q)}_b(\Omega)$ is the subspace of $W^{1,q)}(\Omega)$ such that $|\nabla u| \in L^{q)}_b(\Omega)$. These space revealed very useful in the study of integrability of non negative Jacobians $J_f \ge 0$ of weakly differentiable mappings. Our aim is to prove the following result from [22], with a different proof based on Theorem 1.1:

Theorem 5.1 *Let $f = (u, v) : \Omega \subset \mathbb{R}^2 \xrightarrow{onto} \Omega' \subset \mathbb{R}^2$ a $W_b^{1,2)}$-homeomorphism that is*

$$\lim_{\varepsilon \to 0} \varepsilon \int_\Omega |Df|^{2-\varepsilon} = 0.$$

Then $f \in (N)$.

Proof. By Theorem 1.1 we know that there exists a sequence f_j of $W^{1,2}$-homeomorphisms $f_j = (u_j, v_j) : \Omega \xrightarrow{\text{onto}} \Omega'$ such that

$$f_j \to f \qquad \textit{uniformly} \tag{5.1}$$

$$Df_j \to Df \qquad \textit{in } L^{2)}(\Omega, \mathbb{R}^2) \tag{5.2}$$

and

$$\int_\Omega \varphi J_{f_j} \to \int_\Omega \varphi J_f \qquad \forall \varphi \in C_0^1. \tag{5.3}$$

Since f_j verify the (N)-condition for any Borel set B we have

$$\int_B J_{f_j} = |f_j(B)|.$$

It is sufficient to prove that

$$\int_U J_f = |f(U)| \tag{5.4}$$

for U subdomain of Ω whose boundary consists of finitely many line segments each of which is parallel to the coordinate axis and for which $f_{|\partial U}$ is absolutely continuous. Since f is a homeomorphism which is absolutely continuous an almost all lines, for any compact $E \subset\subset \Omega$ we can always choose such a domain U containing E. We recall Green formula for $u, v \in W^{1,1}(\Omega) \cap C^0(\Omega)$ see

$$\int_A \frac{\partial u}{\partial z} + \frac{\partial v}{\partial \overline{z}} = \frac{\imath}{2} \int_{\partial A} u \mathrm{d}\overline{z} - v \mathrm{d}z \tag{5.5}$$

for $A \subset \Omega$ open set (see [2] pages 35 or [24] pages 150). To prove (5.4), using (5.5), we notice that

$$|f(U)| = |V| = \int_V 1 = \frac{1}{2\imath} \int_{\partial V} \overline{z} \mathrm{d}z = \frac{1}{2\imath} \int_{\partial U} \overline{f} \mathrm{d}f$$

by change of variable in one dimensional integrals which is allowed because f is an absolutely continuous homeomorphism on ∂U. Next obviously we have

$$\frac{1}{2\imath} \int_{\partial U} \overline{f} \mathrm{d}f = \frac{1}{2\imath} \int_{\partial U} \overline{f} f_z \mathrm{d}z + \overline{f} f_{\overline{z}} \mathrm{d}\overline{z}.$$

Moreover let us show that under our assumption we have

$$\frac{1}{2\imath} \int_{\partial U} \overline{f} f_z \mathrm{d}z + \overline{f} f_{\overline{z}} \mathrm{d}\overline{z} = \int_U J_f. \tag{5.6}$$

This equality follows by Green formula for approximating homeomorphism $f_j = (u_j, v_j)$ which belongs to $W^{1,2}$; in fact

$$\int_{\partial U} u_j \mathrm{d}v_j = \iint_U \left(\frac{\partial u_j}{\partial x} \frac{\partial v_j}{\partial y} - \frac{\partial u_j}{\partial y} \frac{\partial v_j}{\partial x} \right) \mathrm{d}x\mathrm{d}y.$$

By (5.3) we deduce

$$\iint_U \left(\frac{\partial u_j}{\partial x} \frac{\partial v_j}{\partial y} - \frac{\partial u_j}{\partial y} \frac{\partial v_j}{\partial x} \right) \mathrm{d}x\mathrm{d}y \to \iint_U \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) \mathrm{d}x\mathrm{d}y.$$

And also by (5.1) we deduce

$$\int_{\partial U} u_j \mathrm{d}v_j \to \int_{\partial U} u \mathrm{d}v$$

which gives (5.4).

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